Difference Operator Approach To Evaluate Integrals Involving Multiple Hypergeometric Functions Of Several Variable With Respect To Parameters

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Abstract

In the present paper, making an appeal to difference operators, we evaluate certain integrals involving multiple hypergeometric functions of Chandel-Gupta (1986), Exton (1972,76), Karlsson (1986), Chandel and Gupa (2007) with respect to parameters. We also apply same technique to evaluate integrals involving hypergeometric functions of four variables due to Sharma and Parihar (1989).

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Keywords: Difference operator E_a , Lauricella's Multiple hypergeometric functions, Appell's hypergeometric functions, Intermediate Lauricella multiple hypergeometric functions due to Karlsson.

Introduction

Recently, making an appeal to difference operator E_a defined by

$$(1.1)\quad E_\alpha f(\alpha)=f(\alpha+1), E_\alpha^n(f(\alpha))=f(\alpha+n),$$

and integral due to Erdélyi [5.p.224]

$$(1.2)\quad \int_{-\infty}^{\infty}\frac{\sin[(m+1)\pi x]dx}{\sin\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}=\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)},\quad Re(\alpha_1+\alpha_2)>1,$$

Joshi and Bhati [, J̃nānāha 27 (1997)] evaluated some integrals involving hypergeometric functions of three and four variables and discussed some special cases.

Recently, making an appeal to difference operators Chandel [2003 presented in ISAAC Congress 2003, York Univ. Toronto Canada] obtained various transformations of multiple hypergeometric functions of several variables due to Chandel-Gupta [, J̃nānāha 16 (1986)], Chandel-Vishwakarma [, J̃nānāha 19 (1989)] and discussed their interesting special cases.

In the present paper, making an appeal to difference operators, we evaluate certain interesting integrals involving multiple hypergeometric functions of several variables $F_A^{(n)}, F_C^{(n)}$ including Intermediate Lauricella's multiple hypergeometric functions and confluent form of Lauricella [16].

2. Techniques Applied. Multiplying both sides of (1.2) by $\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}$

and operating it by the operator

$$\exp\left[u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\right],$$

we have

$$\int_{-\infty}^{\infty} exp\ u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\ \frac{sin(2n'+1)\pi x}{sin\ \pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}dx$$

$$=exp\ u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\}$$

L.H.S.= $\int_{-\infty}^{\infty}\frac{\sin(2m+1)\pi x}{\sin\pi x}\exp\left(u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\right)$

$$\left\{\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\}dx$$

$$=\int_{-\infty}^{\infty}\frac{\sin(2m+1)\pi x}{\sin\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}$$

$$\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{\Gamma(\alpha_1+\ldots+\alpha_n+m_1+\ldots+m_n)}{\Gamma(\alpha_1+m_1+x)\Gamma(\alpha_2+m_2-x)\Gamma(\alpha_3+m_3)\ldots\Gamma(\alpha_n+m_n)}\frac{u_1^{m_1}}{m_1!}\ldots\frac{u_n^{m_n}}{m_n!}dx$$

$$=\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)\Gamma(\alpha_1+\alpha_2-1)}\int_{-\infty}^{\infty}\frac{\sin(2m+1)\pi x}{\sin\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}$$

$$\psi_2^{(n)}(\alpha_1+\ldots+\alpha_n;\alpha_2+x,\alpha_1-x,\alpha_3,\ldots,\alpha_n;u_1,\ldots,u_n)$$

R.H.S.= $\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{u_1^{m_1}}{m_1!}\ldots\frac{u_n^{m_n}}{m_n!}E_{\alpha_1}^{m_1}\ldots E_{\alpha_n}^{m_n}\left\{\frac{2^{\alpha_1+\alpha_2-2}\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\}$

$$=2^{\alpha_1+\alpha_2-2}\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{u_1^{m_1}}{m_1!}\ldots\frac{u_n^{m_n}}{m_n!}\frac{\Gamma(\alpha_1+\ldots+\alpha_n+m_1+\ldots+m_n)}{(\alpha_1+\alpha_2-1)_{m_1+m_2}}\frac{2^{m_1+m_2}}{(\alpha_3)_{m_3}\ldots(\alpha_n)_{m_n}}$$

$$=\frac{2^{\alpha_1+\alpha_2-2}\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1+\alpha_2-1)\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}$$

$$\psi_2^{(n-1)}(\alpha_1+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n)$$

Thus equating L.H.S. and R.H.S., we derive

(2.1.) $\int_{-\infty}^{\infty}\frac{\sin(2m+1)\pi x}{\sin\pi x\Gamma(\alpha_1+x)\Gamma(\alpha_2-x)}$

$$\psi_2^{(n)}(\alpha_1+\ldots+\alpha_n;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,\ldots,u_n)dx$$

$$=\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\psi_2^{(n-1)}(\alpha_1+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n)$$

where $\mathrm{Re}(\alpha_1+\alpha_2)>1, m$ is integer and $\psi_2^{(n)}$ is confluent hypergeometric form of Lauricella's multiple hypergeometric function [16].

For brevity, we consider the integral operator

(2.2)
$$S\{\ \} = \frac{\Gamma(\alpha_1+\alpha_2-1)}{2^{\alpha_1+\alpha_2-2}}\int_{-\infty}^{\infty}\frac{\sin(2m+1)\pi x}{\sin\pi x\,\Gamma(\alpha_1+x)\,\Gamma(\alpha_2-x)}\{\ \}$$

where *m* is an integer and $\mathrm{Re}(\alpha_1+\alpha_2)>1.$

Therefore,

(2.3) $S\{1\}=1,$

and (2.1) can be written as

(2.4)
$$S\left\{\psi_2^{(n)}(\alpha_1+\ldots+\alpha_n;\alpha_1+x,\alpha_2-x,\alpha_3,..,\alpha_n;u_1,..,u_n)\right\}$$
$$=\psi_2^{(n-1)}(\alpha_1+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n),$$
$$\mathrm{Re}(\alpha_1+\alpha_2)>1$$

Considering

$$(1-u_1E_{\alpha_1})^{-\beta_1}\ldots(1-u_nE_{\alpha_n})^{-\beta_n}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\right\}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)},$$

we derive

(2.5)

$$={}^{(2)}F_{AD}^{(n)}(\alpha_1+\ldots+\alpha_n,\beta_1+\ldots+\beta_n;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2u_1,2u_2,u_3,\ldots,u_n),$$

$$Re(\alpha_1+\alpha_2)>1,\ Re(\alpha_i)>0, i=3,\ldots,n,\ |u_1|+\ldots+|u_n|<1,$$

$F_A^{(n)}$ is Lauricella's multiple hypergeometric function [16] and ${}^{(2)}F_{AD}^{(n)}$ is Intermediate Lauricella's multiple hypergeometric function due to Chandel-Gupta [3] for *k=2*.

Similarly,Considering

$$(1-u_1E_{\alpha_1}-u_2E_{\alpha_2})^{-\beta}(1-u_3E_{\alpha_3})^{-\beta_3}\ldots(1-u_nE_{\alpha_n})^{-\beta_n}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we derive

(2.6) $$S\left\{{}^{(2)}F_{AC}^{(n)}(\alpha_1+\ldots+\alpha_n,\beta,\beta_3\ldots\beta_n;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,..,u_n)\right\}$$

$$=F_A^{(n-1)}(\alpha_1+\alpha_2+\ldots+\alpha_n,\beta,\beta_3+\ldots+\beta_n;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;u_1+u_2,u_3,\ldots,u_n),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1, \mathrm{Re}(\alpha_i)>0, i=3,\ldots,n,;|u_1+u_2|+|u_3|+\ldots+|u_4|<1,$ and ${}^{(2)}F_{AC}^{(n)}$ is intermediate Lauricella's multiple hypergeometric function of Chandel-Gupta [3] for *k=2.*

Further considering

$$\left(1-u_1E_{\alpha_1}\right)^{-\beta_1}\left(1-u_2E_{\alpha_2}\right)^{-\beta_2}\ {}_0F_1\left[-;\beta;u_3E_{\alpha_3}+\ldots+u_nE_{\alpha_n}\right]$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}{}\right\},$$

we obtain

(2.7) $$S\left\{{}^{(n-2)}F_{AD}^{(n)}(\alpha_1+\ldots+\alpha_n,\alpha_3,\ldots,\alpha_n,\beta_1,\beta_2,\beta;\alpha_1+x,\alpha_2-x;u_3,\ldots,u_{n-2},u_1,u_2)\right\}$$

$$={}^{(2)}_{(1)}F_D^{(n)}(\alpha_1+\ldots+\alpha_n,\beta_1,\beta_2,\alpha_3+\ldots+\alpha_n;\alpha_1+\alpha_2-1,\beta;2u_1,2u_2,u_3,\ldots,u_n),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$, $\mathrm{Re}(\alpha_i)>0, i=3,\ldots,n,$ and ${}^{(2)}_{(1)}E_D^{(n)}$ is multiple hypergeometric

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we again derive (2.5) specially for $\beta_1=\alpha_1,\beta_2=\alpha_2$.

Further considering

$$\left[1-\left(u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\right)\right]^{-\alpha}\left[1-\left(u_{k+1}E_{\alpha_{k+1}}E_{\beta_{k+1}}+\ldots+u_nE_{\alpha_n}E_{\beta_n}\right)\right]^{-\beta}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{}\right\},$$

and choosing *k=2*, we finally establish

(2.8) $$S\{F_2(\alpha,\alpha_1,\alpha_2;\alpha_1+x,\alpha_2-x;u_1,u_2)\}$$

$$=F_1(\alpha,\alpha_1,\alpha_2;\alpha_1+\alpha_2-1;2u_1,2u_2),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$ and F_1, F_2 are Appell's hypergeometric functions of two variables [1], $max\,(|u_1|,|u_2|)<1/2$,

which has also been obtained by Joshi and Bhati [14, (3.1)] using other operators.

If we consider

$$\left[1-\left(u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\right)\right]^{-\beta}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we obtain

(2.9) $$S\left\{F_C^{(n)}(\alpha_1+\ldots+\alpha_n,\beta;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,\ldots,u_n)\right\}$$

$$=F_C^{(n-1)}(\alpha_1+\ldots+\alpha_n,\beta;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$, $\mathrm{Re}(\alpha_i)>0, i=3,\ldots,n$, $|u_1|^{1/2}+\ldots+|u_n|^{1/2}<1$ and $F_C^{(n)}$ is Lauricella's multiple hypergeometric function of several variables [16].

Further Considering

$$\left(1-u_1E_{\alpha_1}\right)^{-\alpha_1}\ldots\left(1-u_nE_{\alpha_n}\right)^{-\alpha_n}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(b_1+\alpha_3)\ldots\Gamma(b_n+\alpha_n)}\right\},$$

we derive

$$(2.10)\quad S\left\{{}_2F_1(a_1,\alpha_1;\alpha_1+x;u_1).{}_2F_1(a_2,\alpha_2;\alpha_2-x,u_2)\right\}$$

$$=F_3(a_1,a_2,\alpha_1,\alpha_2;\alpha_1+\alpha_2;2u_1,2u_2),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$, $max\,(|u_1|, |u_2|)<1/2$ and F_3 is Appell's function of two variables [1], which also suggests that

$$(2.11)\quad S\left\{{}_1F_1(\alpha_1;\alpha_1+x;u_1).{}_1F_1(\alpha_2;\alpha_2-x;u_2)\right\}$$

$$=\Xi_2(\alpha_1,\alpha_2;\alpha_1+\alpha_2;2u_1,2u_2),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$, $|u_1|<1/2, |u_n|<\infty$ and X_2 is confluent form due to Humbert [12] of Appell's function [1].

Again Considering

$$1-\left(u_1E_{\alpha_1}+\ldots+u_nE_{\alpha_n}\right)^{-\alpha}S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{1}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we arrive at

$$(2.12)\quad S\left\{\psi_2^{(n)}(\alpha;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,u_2,\ldots,u_n)\right\}$$

$$=\psi_2^{(n-1)}(\alpha;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n),$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{AC}\left(\alpha;\alpha_1+\ldots+\alpha_k;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n\right),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1, \mathrm{Re}(\alpha_i)>0, i=3,\ldots,n$ and ${}^{(k)}_{(1)}\phi^{(n)}_{AC}$ is confluent form of intermediate Lauricella's function due to Chandel-Gupta [3].

If we consider

$$\left(1-u_1E_{\alpha_1}\right)^{-b_1}\left(1-u_kE_{\alpha_k}\right)^{-b_k}\left[1-\left(u_{k+1}E_{\alpha_{k+1}}+\ldots+u_nE_{\alpha_n}\right)\right]^{-a}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we derive

$$\text{(2.13)}\quad S\left\{{}^{(k)}_{(1)}\phi^{(n)}_{AC}(\alpha;\alpha_1+\ldots+\alpha_k;\alpha_1+x,\alpha_2-x,\alpha_3,\ldots,\alpha_n;u_1,\ldots,u_n)\right\}$$

$$= {}^{(k)}_{(1)}\phi^{(n)}_{AC}\left(\alpha;\alpha_1+\ldots+\alpha_k;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2(u_1+u_2),u_3,\ldots,u_n\right),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1, \mathrm{Re}(\alpha_i)>0, i=3,\ldots,n$ and ${}^{(k)}_{(1)}\phi^{(n)}_{AC}$ is confluent form of intermediate Lauricella's function due to Chandel-Gupta [3].

If we consider

$$\left(1-u_1E_{\alpha_1}\right)^{-b_1}\left(1-u_kE_{\alpha_k}\right)^{-b_k}\left[1-\left(u_{k+1}E_{\alpha_{k+1}}+\ldots+u_nE_{\alpha_n}\right)\right]^{-a}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\alpha_2+\ldots+\alpha_n)}{\Gamma(\alpha_3)\ldots\Gamma(\alpha_n)}\right\},$$

we arrive at for *k=2*

$$\text{(2.14)}\quad S\left\{{}^{(n-2)}F^{(n)}_{AC}(\alpha_1+\ldots+\alpha_n,a,b_1,b_2;\alpha_3,\ldots,\alpha_n,\alpha_1+x,\alpha_2-x;u_3,\ldots,u_n,u_1,u_2)\right\}$$

$$= {}^{(2)}F^{(n)}_{CD}\left(\alpha_1+\ldots+\alpha_n,a,b_1,b_2;\alpha_1+\alpha_2-1,\alpha_3,\ldots,\alpha_n;2u_1,2u_2,u_3,\ldots,u_n\right),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1, \mathrm{Re}(\alpha_i)>0, i=3,\ldots,n;$ ${}^{(k)}F^{(n)}_{AC}$ is intermediate Lauricella multiple hypergeometric function due to Chandel Gupta [3] while ${}^{(k)}F^{(n)}_{CD}$ is intermediate Lauricella multiple hypergeometric function due to Karlsson [15].

Further Considering

$$\left[1-\left(u_1 E_{\alpha_1}+\ldots+u_k E_{\alpha_k}\right)\right]^{-\alpha}\left(1-u_{k+1} E_{\alpha_{k+1}}\right)^{-a_{k+1}}\left(1-u_k E_{\alpha_k}\right)^{-a_n}$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(\alpha_3+\ldots+\alpha_n)}\right\},$$

we finally derive for *k=2*

$$S\left\{{}^{(n-2)}F_{CD}^{(n)}(\alpha_1+\ldots+\alpha_n,a,a_{n-1},a_n;\alpha_3,\ldots,\alpha_n,\alpha_1+x,\alpha_2-x;u_{n-1},u_n,u_1\ldots,u_{n-2})\right\} \quad (2.15)$$

$$={}^{(n-2)}F_{CD}^{(n-1)}(\alpha_1+\alpha_2+\ldots+\alpha_n,a,a_{n-1},a_n;\alpha_3,\ldots,\alpha_n,\alpha_1+\alpha_2-1;u_3,\ldots,u_n,2(u_1+u_2)),$$

$\mathrm{Re}(\alpha_1+\alpha_2)>1$, $\mathrm{Re}(\alpha_3+\ldots+\alpha_n)>0$ and ${}^{(k)}F_{CD}^{(n)}$ is intermediate Lauricella's function due to Karlsson [15].

If we consider

$$exp\left(u_1 E_{\alpha_3} E_{\alpha'_3}+u_2 E_{\alpha_1}+u_3 E_{\alpha_2}+u_4 E_{\alpha_4} E_{\alpha'_4}\right)$$

$$S\left\{\frac{2^{\alpha_1+\alpha_2-2}}{\Gamma(\alpha_1+\alpha_2-1)}\frac{\Gamma(\alpha_1+\alpha_3)\Gamma(\alpha_2+\alpha_4)\Gamma(\alpha_2+\alpha_3)\Gamma(\alpha_1+\alpha_4)}{\Gamma(\alpha'_3+\alpha'_4)}\right\},$$

we derive

$$S\left\{F_{29}^{(4)}(\alpha_1+\alpha_3,\alpha_1+\alpha_3,\alpha_2+\alpha_4,\alpha_2+\alpha_4,\alpha_2+\alpha_3,\alpha_1+\alpha_4,\alpha_2+\alpha_3,\alpha_1+\alpha_4;\right. \quad (2.16)$$

$$\left.\alpha'_3+\alpha'_4,\alpha_1+x,\alpha_2-x,\alpha'_3+\alpha'_4;u_1,u_2,u_3,u_4)\right\}$$

$$=F_{58}^{(4)}(\alpha_1+\alpha_3,\alpha_1+\alpha_3,\alpha_2+\alpha_4,\alpha_2+\alpha_4,\alpha_2+\alpha_3,\alpha_1+\alpha_4,\alpha_2+\alpha_3,\alpha_1+\alpha_4;$$

$$\alpha'_3+\alpha'_4,\alpha_1+\alpha_2-1,\alpha_1+\alpha_2-1,\alpha'_3+\alpha'_4;u_1,2u_2,2u_3,u_4),$$

$\mathrm{Re}(\alpha_1'+\alpha_2)>0, \mathrm{Re}(\alpha_i)>0, i=3,4$ and $F_{29}^{(4)}, F_{58}^{(4)}$ are hypergeometric functions of four variables defined by Sharma and Parihar [17].

Making similar difference operational approach, several other interesting integrals involving multiple hypergeometric functions of different variables can be evaluated.

Objective of the Study

Making an appeal to difference operators, we want to evacuate certain integers involving multiple hypergeometric functions.

Conclusion

In the present paper, making an appeal to difference operators, we evaluate certain interesting integrals involving multiple hypergeometric functions of several variables including intermediate lauricella's multiple hypergeometric functions and confluent form of lauricella.

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